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Clifford algebra-valued orthogonal polynomials in Euclidean space

F. Brackx, N. De Schepper*, F. Sommen

Clifford Research Group, Department of Mathematical Analysis, Faculty of Engineering, Ghent University, Galglaan 2, 9000 Gent, Belgium

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Abstract

In this paper a new method for constructing Clifford algebra-valued orthogonal polynomials in Euclidean space is presented. In earlier research, only scalar-valued weight functions were involved. Now the class of weight functions is enlarged with Clifford algebra-valued functions.

The method consists in transforming the orthogonality relation on the Euclidean space into an orthogonality relation on the real axis by means of the so-called Clifford–Heaviside functions. Consequently appropriate orthogonal polynomials on the real axis yield Clifford algebra-valued orthogonal polynomials in Euclidean space.

Three specific examples of such orthogonal polynomials in Euclidean space are discussed, viz. the generalized Clifford–Hermite, the Clifford–Laguerre and the half-range Clifford–Hermite polynomials.
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1. Introduction

In a series of papers [4–6,8,9] higher-dimensional wavelets and their corresponding continuous wavelet transforms have been studied within the framework of Clifford analysis. Clifford analysis, centred around the notion of monogenic function, may be regarded as a direct and elegant gen-

* Corresponding author.
E-mail addresses: fb@cage.ugent.be (F. Brackx), nds@cage.ugent.be (N. De Schepper), fs@cage.ugent.be (F. Sommen).

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eralization to higher dimension of the theory of holomorphic functions in the complex plane. An essential step in the construction of those Clifford-wavelets is the introduction of specific polynomials satisfying orthogonality relations with respect to scalar-valued weight functions. These polynomials originate as a result of a particular Clifford analysis technique, the so-called Cauchy–Kowalewskaia extension of a real-analytic function in $\mathbb{R}^m$ to a monogenic function in $\mathbb{R}^{m+1}$. The Clifford–Hermite, Clifford–Gegenbauer and Clifford–Laguerre polynomials constructed this way, all give rise to wavelets in $\mathbb{R}^m$ since they all satisfy the necessary admissibility condition. The respective orthogonality relations lead to a number of vanishing moments, an important feature in wavelet theory. For an account of the continuous wavelet transform in Clifford analysis and an overview of the generalized orthogonal polynomials and their corresponding wavelets thus far obtained, we refer the reader to [7].

In our quest for new Clifford wavelets we came across a simple but highly efficient method for constructing Clifford algebra-valued orthogonal polynomials in Euclidean space. It should be emphasized that the class of weight functions, which up to now always were scalar-valued, is now enlarged with Clifford algebra-valued real-analytic functions. Unfortunately the newly obtained orthogonal polynomials fail to satisfy the necessary admissibility condition in order to make them candidates for a mother wavelet. However both the method and the higher-dimensional orthogonal polynomials it generates, have a value of their own.

In order to make the paper self-contained, a section on definitions and basic properties of Clifford algebra and Clifford analysis is included (see Section 2).

In Section 3 our methodology is presented. It consists, roughly speaking, of transforming the orthogonality relation on the Euclidean space into an orthogonality relation on the real axis. Crucial to this transformation are the so-called Clifford–Heaviside functions; they generalize to higher dimension the Heaviside step-function on the real axis and are a typical feature of Clifford analysis. Apparently our construction method is simple, but nevertheless it should be emphasized that this is entirely due to the power of Clifford analysis and the existence of these idempotent Clifford–Heaviside functions, inexisting in complex or harmonic analysis.

The method is then applied to three specific cases; in each of the cases known orthogonal polynomials on the interval $]-\infty, +\infty[$ or $[0, +\infty[$ lead to orthogonal Clifford algebra-valued polynomials in Euclidean space. The obtained Clifford–Laguerre (see Section 5) and half-range Clifford–Hermite polynomials (see Section 6) are entirely new; the generalized Clifford–Hermite polynomials (see Section 4) coincide, up to constants, with the radial Clifford–Hermite polynomials, already introduced in [19] by means of the Cauchy–Kowalewskaia extension. A number of those higher-dimensional orthogonal polynomials is explicitly calculated and in each case an explicit recurrence relation is established.

2. Clifford algebra and Clifford analysis

Clifford analysis (see e.g. [2,12]) offers a function theory which is a higher-dimensional analogue of the theory of the holomorphic functions of one complex variable. The functions considered are defined in $\mathbb{R}^m$ $(m > 1)$ and take their values in the Clifford algebra $\mathbb{R}_m$ or its complexification $\mathbb{C}_m$. If $(e_1, \ldots, e_m)$ is an orthonormal basis of $\mathbb{R}^m$, then a basis for the Clifford algebra $\mathbb{R}_m$ is given by $(e_A : A \subset \{1, \ldots, m\})$ where $e_\emptyset = 1$ is the identity element. The non-commutative multiplication in the Clifford algebra is governed by the rules:

$$\begin{aligned} e_j^2 &= -1, \quad j = 1, \ldots, m, \\ e_j e_k + e_k e_j &= 0, \quad j \neq k, \quad j, k = 1, \ldots, m. \end{aligned}$$

Conjugation is defined as the anti-involution for which

$$\overline{e}_j = -e_j, \quad j = 1, \ldots, m,$$

with the additional rule $\overline{i} = -i$ in the case of $\mathbb{C}_m$.

For $k = 0, 1, \ldots, m$ fixed, we call

$$\mathbb{C}_m^k = \left\{ \sum_{\#A=k} a_A e_A;\ a_A \in \mathbb{C} \right\}$$

the subspace of k-vectors, i.e. the space spanned by the products of k different basis vectors.

The Euclidean space $\mathbb{R}^m$ is embedded in the Clifford algebras $\mathbb{R}_m$ and $\mathbb{C}_m$ by identifying $(x_1, \ldots, x_m)$ with the vector variable $\underline{x}$ given by

$$\underline{x} = \sum_{j=1}^{m} e_j x_j.$$

The product of two vectors splits up into a scalar part and a 2-vector, also called bivector, part:

$$\underline{x}\,\underline{y} = \underline{x} \cdot \underline{y} + \underline{x} \wedge \underline{y},$$

where

$$\underline{x} \cdot \underline{y} = -\langle \underline{x}, \underline{y} \rangle = -\sum_{j=1}^{m} x_j y_j$$

and

$$\underline{x} \wedge \underline{y} = \sum_{j=1}^{m} \sum_{k=j+1}^{m} e_j e_k (x_j y_k - x_k y_j).$$

In particular

$$\underline{x}^2 = -\langle \underline{x}, \underline{x} \rangle = -|\underline{x}|^2.$$

The Spin-group

$$Spin(m) = \{ s = \underline{\omega}_1 \ldots \underline{\omega}_{2\ell};\ \underline{\omega}_j \in S^{m-1},\ j = 1, \ldots, 2\ell,\ \ell \in \mathbb{N} \},$$

where S^{m-1} denotes the unit sphere in $\mathbb{R}^m$, is a two-fold covering group of the rotation group $SO(m)$. For $T \in SO(m)$, there exists $s \in Spin(m)$ such that $T(\underline{x}) = s\underline{x}\overline{s} = (-s)\underline{x}(-\overline{s})$, for all $\underline{x} \in \mathbb{R}^m$.

In the sequel, the so-called Clifford–Heaviside functions

$$P^+ = \frac{1}{2}\left(1 + i\,\frac{\underline{x}}{|\underline{x}|}\right), \quad P^- = \frac{1}{2}\left(1 - i\,\frac{\underline{x}}{|\underline{x}|}\right)$$

will play an important rôle; they were introduced independently by Sommen [18] and McIntosh [16].

Introducing spherical co-ordinates in $\mathbb{R}^m$ by

$$\underline{x} = r\underline{\omega}, \quad r = |\underline{x}| \in [0, +\infty[, \quad \underline{\omega} \in S^{m-1},$$

the Clifford–Heaviside functions can be rewritten as

$$P^+ = \frac{1}{2}\left(1 + i\underline{\omega}\right), \quad P^- = \frac{1}{2}\left(1 - i\underline{\omega}\right).$$

They satisfy the relations

$$P^+ + P^- = 1; \quad P^+P^- = P^-P^+ = 0, \quad (P^+)^2 = P^+, \quad (P^-)^2 = P^-.$$

Furthermore, we have

$$i\underline{\omega}\ P^\pm = \pm\ P^\pm \quad \text{and hence } i\underline{x}\ P^\pm = \pm\, r\ P^\pm.$$

The central notion in Clifford analysis is the notion of monogenicity, the higher-dimensional analogue of holomorphicity.

An $\mathbb{R}_m$- or $\mathbb{C}_m$-valued function $F(x_1, \ldots, x_m)$ is called left monogenic in an open region of $\mathbb{R}^m$, if in that region:

$$\partial_{\underline{x}} F = 0.$$

Here $\partial_{\underline{x}}$ is the Dirac operator in $\mathbb{R}^m$:

$$\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial_{x_j},$$

a vector elliptic differential operator of the first-order splitting the Laplacian in $\mathbb{R}^m$:

$$\Delta_m = -\partial_{\underline{x}}^2.$$

The notion of right monogenicity is defined in a similar way by letting act the Dirac operator from the right.

Let $\Omega \subset \mathbb{R}^m$ be open, let C be a compact orientable m-dimensional manifold with boundary ∂C and define the oriented $\mathbb{R}_m$-valued surface element $d\sigma_{\underline{x}}$ on ∂C by

$$d\sigma_{\underline{x}} = \sum_{j=1}^{m} (-1)^j e_j\, d\widehat{x}_j,$$

where

$$d\widehat{x}_j = dx_1 \wedge \cdots \wedge [dx_j] \wedge \cdots \wedge dx_m, \quad j = 1, 2, \ldots, m.$$

If $n(\underline{x})$ stands for the outward pointing unit normal at $\underline{x} \in \partial C$, then

$$d\sigma_{\underline{x}} = n(\underline{x})\, d\Sigma(\underline{x}),$$

$d\Sigma(\underline{x})$ being the Lebesgue surface measure.

Suppose that $f \in C_1(\Omega)$ is right monogenic in Ω. Then Cauchy's Theorem states that for each $C \subset \Omega$,

$$\int_{\partial C} f(\underline{x})\, d\sigma_{\underline{x}} = 0.$$

An important particular example occurs in the following case: take $f = 1$ and $C = \overline{B}(1) = \{\underline{x} \in \mathbb{R}^m;\, |\underline{x}| \leqslant 1\}$, the closed unit ball in $\mathbb{R}^m$. Then $\partial C = S^{m-1}$ and at each point $\underline{\omega} \in S^{m-1}$, $n(\underline{\omega}) = \underline{\omega}$, whence $d\sigma_{\underline{\omega}} = \underline{\omega}\, dS(\underline{\omega})$ with $dS(\underline{\omega})$ the Lebesgue measure on S^{m-1}.

Consequently, we have

$$\int_{S^{m-1}} \underline{\omega}\, dS(\underline{\omega}) = 0,$$

confirming the fact that $\underline{\omega}$ is a spherical harmonic.

The above result will be of crucial importance in our general method for constructing Clifford algebra-valued orthogonal polynomials in Euclidean space.

3. The general construction method

In this section we expose our methodology for constructing Clifford algebra-valued polynomials of the form

$$p_n(i\underline{x}) = \sum_{k=0}^{n} a_k (i\underline{x})^k, \quad a_k \in \mathbb{C}, \quad k = 0, 1, 2, \ldots, n,$$

which are orthogonal on the Euclidean space $\mathbb{R}^m$ with respect to a Clifford algebra-valued weight function. Note that the polynomials considered take their values in $\mathbb{C}_m^0 \oplus \mathbb{C}_m^1$, i.e. a scalar plus a vector, also called paravector.

Definition. If $W(r) = \sum_{j=0}^{\infty} b_j r^j$ ($b_j \in \mathbb{C}$, $j \in \mathbb{N} \cup \{0\}$) is real-analytic in the neighbourhood of the origin $r = 0$, then one defines $W(i\underline{x}) = \sum_{j=0}^{\infty} b_j (i\underline{x})^j$.

Proposition. *If $W(r)$ is real-analytic in $]-\rho, \rho[$ then in $\overset{o}{B}(0, \rho) = \{\underline{x} \in \mathbb{R}^m;\, |\underline{x}| < \rho\}$ one has:*
(i) *$W(i\underline{x})$ is real-analytic in the variables $(x_1, \ldots, x_m)$,*
(ii) $W(i\underline{x})P^+ = P^+W(i\underline{x}) = W(r)P^+$,
(iii) $W(i\underline{x})P^- = P^-W(i\underline{x}) = W(-r)P^-$,
(iv) $W(i\underline{x}) = W(r)P^+ + W(-r)P^-$.

Proof.
(i) Straightforward.
(ii) Applying the properties of P^+, we have successively

$$W(i\underline{x})P^+ = \sum_{j=0}^{\infty} b_j (i\underline{x})^j P^+ = \sum_{j=0}^{\infty} b_j (i\underline{x})^j (P^+)^j = \sum_{j=0}^{\infty} b_j (rP^+)^j = \sum_{j=0}^{\infty} b_j r^j P^+$$

and thus

$$W(i\underline{x})P^+ = W(r)P^+.$$

Moreover

$$W(i\underline{x})P^+ = P^+W(i\underline{x}).$$

(iii) Similar to (ii).
(iv) The formulae in (ii) and (iii) lead to

$$\begin{aligned} W(i\underline{x}) &= W(i\underline{x})P^+ + W(i\underline{x})P^- \\ &= W(r)P^+ + W(-r)P^-, \end{aligned}$$

where we have used the fact that $P^+ + P^- = 1$. □

In what follows we will show how, by means of the above properties, integrals over the Euclidean space $\mathbb{R}^m$ can be rewritten in terms of integrals over the real axis. Consequently, constructing Clifford algebra-valued polynomials $\{p_n(i\underline{x})\}_{n\geqslant 0}$ which are orthogonal on $\mathbb{R}^m$ will be reduced to constructing orthogonal polynomials on the real axis.

Two types of Clifford algebra-valued orthogonal polynomials on $\mathbb{R}^m$ will be distinguished:

3.1. *Type 1: orthogonality on $\mathbb{R}^m$ with respect to the weight function $W(i\underline{x})$*

First we search for Clifford algebra-valued polynomials $\{p_n(i\underline{x})\}_{n\geqslant 0}$ which are orthogonal on $\mathbb{R}^m$ with respect to a Clifford algebra-valued weight function $W(i\underline{x})$, thus satisfying the orthogonality relation

$$\Big(p_n(i\underline{x}),\, p_{n'}(i\underline{x})\Big) = \int_{\mathbb{R}^m} \overline{p}_n(i\underline{x})W(i\underline{x})p_{n'}(i\underline{x})\,dV(\underline{x}) = 0,$$

whenever $n \neq n'$. Here $dV(\underline{x})$ stands for the Lebesgue measure on $\mathbb{R}^m$.

Using the fact that $P^+ + P^- = 1$, we have

$$\begin{aligned} \Big(p_n(i\underline{x}),\, p_{n'}(i\underline{x})\Big) &= \int_{\mathbb{R}^m} \overline{p}_n(i\underline{x})W(i\underline{x})p_{n'}(i\underline{x})P^+\,dV(\underline{x}) \\ &\quad + \int_{\mathbb{R}^m} \overline{p}_n(i\underline{x})W(i\underline{x})p_{n'}(i\underline{x})P^-\,dV(\underline{x}). \end{aligned} \tag{1}$$

As the polynomials $\{p_n(i\underline{x})\}_{n\geqslant 0}$ satisfy

$$p_n(i\underline{x})P^+ = p_n(r)P^+ \qquad p_n(i\underline{x})P^- = p_n(-r)P^-$$

and taking into account that (see Section 2)

$$\int_{S^{m-1}} \underline{\omega}\,dS(\underline{\omega}) = 0,$$

expression (1) can be simplified to:

$$\begin{aligned} &\Big(p_n(i\underline{x}),\, p_{n'}(i\underline{x})\Big) \\ &\quad = \int_{\mathbb{R}^m} \overline{p}_n(r)W(r)p_{n'}(r)P^+dV(\underline{x}) + \int_{\mathbb{R}^m} \overline{p}_n(-r)W(-r)p_{n'}(-r)P^-dV(\underline{x}) \\ &\quad = \int_0^{+\infty} \overline{p}_n(r)W(r)p_{n'}(r)r^{m-1}\,dr \int_{S^{m-1}} \frac{1}{2}(1+i\underline{\omega})\,dS(\underline{\omega}) \end{aligned}$$

$$
\begin{aligned}
&+\int_0^{+\infty}\overline{p}_n(-r)W(-r)p_{n'}(-r)r^{m-1}\,dr\int_{S^{m-1}}\frac{1}{2}(1-i\underline{\omega})\,dS(\underline{\omega})\\
&=\frac{A_m}{2}\left(\int_0^{+\infty}\overline{p}_n(r)W(r)p_{n'}(r)r^{m-1}\,dr\right.\\
&\quad\left.+\int_0^{+\infty}\overline{p}_n(-r)W(-r)p_{n'}(-r)r^{m-1}\,dr\right)\\
&=\frac{A_m}{2}\left(\int_0^{+\infty}\overline{p}_n(r)W(r)p_{n'}(r)|r|^{m-1}\,dr+\int_{-\infty}^{0}\overline{p}_n(u)W(u)p_{n'}(u)|u|^{m-1}du\right)\\
&=\frac{A_m}{2}\int_{-\infty}^{+\infty}\overline{p}_n(r)W(r)|r|^{m-1}p_{n'}(r)\,dr.
\end{aligned}
$$

Here A_m denotes the area of the unit sphere S^{m-1} in $\mathbb{R}^m$.

So we may conclude that the polynomials $\{p_n(i\underline{x})\}_{n\geqslant 0}$ are orthogonal on $\mathbb{R}^m$ with respect to $W(i\underline{x})$ if and only if the polynomials $\{p_n(r)\}_{n\geqslant 0}$ are orthogonal on $]-\infty,+\infty[$ with respect to the weight function $W(r)|r|^{m-1}$. Hereby it is tacitly assumed that the weight functions $W(r)$ and $W(i\underline{x})$ are real-analytic in, respectively, $]-\infty,+\infty[$ and $\mathbb{R}^m$ and that moreover all integrals involved are convergent.

Note that in the special case where the dimension m is odd, the polynomials $\{p_n(r)\}_{n\geqslant 0}$ should be orthogonal on $]-\infty,+\infty[$ with respect to the weight function $W(r)r^{m-1}$.

3.2. *Type 2: orthogonality on $\mathbb{R}^m$ with respect to the weight function $W(i\underline{x})P^+$*

Now we consider the construction of Clifford algebra-valued polynomials $\{p_n(i\underline{x})\}_{n\geqslant 0}$ which satisfy the orthogonality relation

$$
\Big(p_n(i\underline{x}),\,p_{n'}(i\underline{x})\Big)=\int_{\mathbb{R}^m}\overline{p}_n(i\underline{x})W(i\underline{x})P^+p_{n'}(i\underline{x})\,dV(\underline{x})=0,
$$

whenever $n\neq n'$; thus we search for orthogonal polynomials $\{p_n(i\underline{x})\}_{n\geqslant 0}$ in $\mathbb{R}^m$ with respect to a Clifford algebra-valued weight function of the form $W(i\underline{x})P^+=W(r)P^+$.

By introducing spherical co-ordinates we obtain:

$$
\begin{aligned}
\Big(p_n(i\underline{x}),\,p_{n'}(i\underline{x})\Big)&=\int_{\mathbb{R}^m}\overline{p}_n(r)W(r)p_{n'}(r)P^+dV(\underline{x})\\
&=\int_0^{+\infty}\overline{p}_n(r)W(r)p_{n'}(r)r^{m-1}\,dr\int_{S^{m-1}}\frac{1}{2}(1+i\underline{\omega})\,dS(\underline{\omega})\\
&=\frac{A_m}{2}\int_0^{+\infty}\overline{p}_n(r)W(r)r^{m-1}p_{n'}(r)\,dr.
\end{aligned}
$$

Consequently, the polynomials $\{p_n(i\underline{x})\}_{n\geqslant 0}$ are orthogonal on $\mathbb{R}^m$ with respect to $W(i\underline{x})P^+$ if and only if the polynomials $\{p_n(r)\}_{n\geqslant 0}$ are orthogonal on $[0,+\infty[$ with respect to the weight function $W(r)r^{m-1}$.

Note that when considering in $\mathbb{R}^m$ the weight function $W(i\underline{x})P^-=W(-r)P^-$, one has in a similar way as above

$$
\Big(p_n(i\underline{x}),\,p_{n'}(i\underline{x})\Big)=\frac{A_m}{2}(-1)^{m-1}\int_{-\infty}^{0}\overline{p}_n(r)W(r)r^{m-1}p_{n'}(r)\,dr.
$$

In this case we thus need polynomials $\{p_n(r)\}_{n\geqslant 0}$ which are orthogonal on $]-\infty, 0]$ with respect to the weight function $W(r)r^{m-1}$.

4. The generalized Clifford–Hermite polynomials

In this section we focus on Clifford algebra-valued orthogonal polynomials in $\mathbb{R}^m$ with respect to the specific weight function $W(i\underline{x}) = \exp(-|i\underline{x}|^2) = \exp(-|\underline{x}|^2)$. Hereby we will follow the general theory of Section 3.1.

The construction is based on the monic generalized Hermite polynomials $K_n^{(\gamma)}(x)$ orthogonal on $]-\infty, +\infty[$ with respect to $|x|^\gamma \exp(-x^2)$; $\gamma > -1$ (see e.g. [10,13,20]).

These polynomials $K_n^{(\gamma)}(x)$ satisfy the recurrence relation

$$K_{n+1}^{(\gamma)}(x) = xK_n^{(\gamma)}(x) - \widehat{a}_n K_{n-1}^{(\gamma)}(x), \quad n\geqslant 0,$$
$$K_{-1}^{(\gamma)}(x) = 0, \quad K_0^{(\gamma)}(x) = 1 \tag{2}$$

with

$$\widehat{a}_n = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n+\gamma}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, they can be expressed in terms of the generalized Laguerre polynomials on the real line:

$$K_{2n}^{(\gamma)}(x) = (-1)^n\, n!\, L_n^{(\gamma/2-1/2)}(x^2),$$
$$K_{2n+1}^{(\gamma)}(x) = (-1)^n\, n!\, x\, L_n^{(\gamma/2+1/2)}(x^2).$$

These generalized Laguerre polynomials $L_n^{(\alpha)}(x)$, for $\alpha > -1$, are orthogonal polynomials associated with the interval $[0, +\infty[$ and the weight function $x^\alpha \exp(-x)$.

From their explicit expression (see for e.g. [15])

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{x^k}{k!}, \tag{3}$$

we obtain

$$K_{2n}^{(\gamma)}(x) = (-1)^n n! \sum_{k=0}^{n} (-1)^k \binom{n+\frac{\gamma}{2}-\frac{1}{2}}{n-k} \frac{x^{2k}}{k!},$$
$$K_{2n+1}^{(\gamma)}(x) = n! \sum_{k=0}^{n} (-1)^{n+k} \binom{n+\frac{\gamma}{2}+\frac{1}{2}}{n-k} \frac{x^{2k+1}}{k!}.$$

The generalized Hermite polynomials $\{K_n^{(m-1)}(x)\}_{n\geqslant 0}$ are orthogonal on $]-\infty, +\infty[$ with respect to $\exp(-x^2)|x|^{m-1}$. According to Section 3.1, these polynomials are the desired building blocks for the Clifford algebra-valued polynomials $\{K_n(i\underline{x})\}_{n\geqslant 0}$, which are orthogonal on $\mathbb{R}^m$ with respect to $\exp(-|\underline{x}|^2)$. We will call these polynomials the generalized Clifford–Hermite polynomials.

Converting the above results for the generalized Hermite polynomials to the Clifford analysis setting according to the general construction method of Section 3, we obtain the recurrence relation

$$K_{n+1}(i\underline{x}) = i\underline{x} K_n(i\underline{x}) - \widetilde{a}_n K_{n-1}(i\underline{x}), \quad n \geqslant 0,$$

$$K_{-1}(i\underline{x}) = 0, \quad K_0(i\underline{x}) = 1,$$

where now

$$\widetilde{a}_n = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n+m-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

The connection with the classical generalized Laguerre polynomials is given by

$$K_{2n}(i\underline{x}) = (-1)^n \; n! \; L_n^{(m/2-1)}(|\underline{x}|^2),$$
$$K_{2n+1}(i\underline{x}) = (-1)^n \; n! \; i\underline{x} \; L_n^{(m/2)}(|\underline{x}|^2)$$

and we have the explicit expression

$$K_{2n}(i\underline{x}) = (-1)^n n! \sum_{k=0}^{n} (-1)^k \binom{n + \frac{m}{2} - 1}{n-k} \frac{(i\underline{x})^{2k}}{k!},$$
$$K_{2n+1}(i\underline{x}) = n! \sum_{k=0}^{n} (-1)^{n+k} \binom{n + \frac{m}{2}}{n-k} \frac{(i\underline{x})^{2k+1}}{k!}.$$

The first generalized Clifford–Hermite polynomials are given by

$$K_0(i\underline{x}) = 1,$$
$$K_1(i\underline{x}) = i\underline{x},$$
$$K_2(i\underline{x}) = (i\underline{x})^2 - \left(\frac{m}{2}\right) = |\underline{x}|^2 - \left(\frac{m}{2}\right),$$
$$K_3(i\underline{x}) = (i\underline{x})^3 - \left(\frac{m+2}{2}\right) i\underline{x} = i|\underline{x}|^2 \underline{x} - \left(\frac{m+2}{2}\right) i\underline{x},$$
$$K_4(i\underline{x}) = (i\underline{x})^4 - (m+2)(i\underline{x})^2 + \left(\frac{m+2}{2}\right)\left(\frac{m}{2}\right) = |\underline{x}|^4 - (m+2)|\underline{x}|^2 + \left(\frac{m+2}{2}\right)\left(\frac{m}{2}\right),$$
$$K_5(i\underline{x}) = (i\underline{x})^5 - (m+4)(i\underline{x})^3 + \left(\frac{m+4}{2}\right)\left(\frac{m+2}{2}\right)(i\underline{x}) = i|\underline{x}|^4 \underline{x} - i(m+4)|\underline{x}|^2 \underline{x} + \left(\frac{m+4}{2}\right)\left(\frac{m+2}{2}\right)(i\underline{x}),$$

etc.

Note that $K_{2n}(i\underline{x})$ is real-valued, while $K_{2n+1}(i\underline{x})$ is complex vector-valued.

It should be noted that Clifford algebra-valued orthogonal polynomials in $\mathbb{R}^m$ with respect to the weight function $\exp\left(-\frac{|\underline{x}|^2}{2}\right)$ were already introduced in [19] by Sommen. These so-called radial Clifford–Hermite polynomials $\{H_{n,m}(\underline{x})\}_{n\geqslant 0}$ were constructed by means of the Cauchy–Kowalewskaia extension.

They can be expressed in terms of the generalized Laguerre polynomials on the real line as follows:

$$\begin{aligned} H_{2n,m}(\underline{x}) &= 2^n n!\, L_n^{(m/2-1)}\left(-\frac{\underline{x}^2}{2}\right), \\ H_{2n+1,m}(\underline{x}) &= 2^n n!\, \underline{x}\, L_n^{(m/2)}\left(-\frac{\underline{x}^2}{2}\right), \end{aligned}$$

which corrects a result from [11, p. 70; 12, p. 309].

Consequently we have

$$\begin{aligned} H_{2n,m}(\sqrt{2}\underline{x}) &= 2^n n!\, L_n^{(m/2-1)}(|\underline{x}|^2), \\ H_{2n+1,m}(\sqrt{2}\underline{x}) &= 2^n n!\, \sqrt{2}\, \underline{x}\, L_n^{(m/2)}(|\underline{x}|^2), \end{aligned}$$

from which we obtain the following relation between the generalized Clifford–Hermite polynomials and the radial Clifford–Hermite polynomials:

$$\begin{aligned} K_{2n}(i\underline{x}) &= (-1)^n 2^{-n} H_{2n,m}(\sqrt{2}\underline{x}), \\ K_{2n+1}(i\underline{x}) &= (-1)^n i\, 2^{-(n+1/2)} H_{2n+1,m}(\sqrt{2}\underline{x}). \end{aligned}$$

5. The Clifford–Laguerre polynomials

In this section we construct Clifford algebra-valued polynomials which are orthogonal on $\mathbb{R}^m$ with respect to the Clifford algebra-valued weight function $W(i\underline{x})P^+ = \exp(-i\underline{x})(i\underline{x})^\alpha P^+$; $\alpha > -m$.

The first factor in this weight function $\exp(-i\underline{x})$ is defined by means of the real-analytic function $\exp(-r)$ on the real r-axis (definition Section 3). The second factor $(i\underline{x})^\alpha$, $\alpha > -m$ is defined by

$$(i\underline{x})^\alpha = r^\alpha\left(P^+ + \exp(i\pi\alpha)P^-\right)$$

(see [12, p. 349; 3, p. 14]).

Note that indeed:

$$(i\underline{x})^\alpha P^+ = r^\alpha\left(\left(P^+\right)^2 + \exp(i\pi\alpha)P^-P^+\right) = r^\alpha P^+$$

and hence

$$\begin{aligned} W(i\underline{x})P^+ &= \exp(-i\underline{x})(i\underline{x})^\alpha P^+ = \exp(-i\underline{x})r^\alpha P^+ \\ &= \exp(-i\underline{x})P^+ r^\alpha = \exp(-r)r^\alpha P^+. \end{aligned}$$

By the change of variables $\alpha \to \alpha + m - 1 > -1$ and $x \to r$ in the orthogonality relation for the generalized Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n \geqslant 0}$, $\alpha > -1$, we get

$$\int_0^{+\infty} \exp(-r) r^{\alpha+m-1} L_n^{(\alpha+m-1)}(r) L_{n'}^{(\alpha+m-1)}(r)\, dr = \Gamma(\alpha+m)\binom{n+\alpha+m-1}{n}\delta_{n,n'}.$$

In a similar way the explicit expression (3) and the recurrence relation for the generalized Laguerre polynomials lead to

$$L_n^{(\alpha+m-1)}(r) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha+m-1}{n-k}\frac{r^k}{k!}$$

and

$$nL_n^{(\alpha+m-1)}(r) = (2n+\alpha+m-2-r)L_{n-1}^{(\alpha+m-1)}(r) - (n+\alpha+m-2)L_{n-2}^{(\alpha+m-1)}(r).$$

The above results for the polynomials $\{L_n^{(\alpha+m-1)}(r)\}_{n \geqslant 0}$ immediately give rise to the corresponding results for the Clifford algebra-valued polynomials $\{L_n^{(\alpha)}(i\underline{x})\}_{n \geqslant 0}$ orthogonal on $\mathbb{R}^m$ with respect to $\exp(-i\underline{x})(i\underline{x})^{\alpha} P^+$, $\alpha > -m$, which we call the Clifford–Laguerre polynomials. They take the explicit form

$$L_n^{(\alpha)}(i\underline{x}) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha+m-1}{n-k}\frac{(i\underline{x})^k}{k!}$$

and satisfy the recurrence relation

$$nL_n^{(\alpha)}(i\underline{x}) = (2n+\alpha+m-2-i\underline{x})L_{n-1}^{(\alpha)}(i\underline{x}) - (n+\alpha+m-2)L_{n-2}^{(\alpha)}(i\underline{x}).$$

The first Clifford–Laguerre polynomials are calculated to be:

$$\begin{aligned}
L_0^{(\alpha)}(i\underline{x}) &= 1,\\
L_1^{(\alpha)}(i\underline{x}) &= -i\underline{x} + \alpha + m,\\
L_2^{(\alpha)}(i\underline{x}) &= \frac{1}{2}(i\underline{x})^2 - (\alpha+m+1)i\underline{x} + \frac{1}{2}(\alpha+m)(\alpha+m+1),\\
L_3^{(\alpha)}(i\underline{x}) &= -\frac{1}{6}(i\underline{x})^3 + \frac{1}{2}(\alpha+m+2)(i\underline{x})^2 - \frac{1}{2}(\alpha+m+1)(\alpha+m+2)(i\underline{x})\\
&\quad + \frac{1}{6}(\alpha+m)(\alpha+m+1)(\alpha+m+2),
\end{aligned}$$

etc.

6. The half-range Clifford–Hermite polynomials

This section contains the construction of Clifford algebra-valued orthogonal polynomials in $\mathbb{R}^m$ with respect to the Clifford algebra-valued weight function $W(i\underline{x})P^+ = \exp(-(i\underline{x})^2)P^+$; they are called the half-range Clifford–Hermite polynomials.

According to the general theory exposed in Section 3.2, this construction is based on orthogonal polynomials on $[0, +\infty[$ with respect to the weight function $\exp(-r^2)r^{m-1}$.

In [1] a method is developed for calculating the coefficients in the recurrence relation for the so-called half-range generalized Hermite polynomials on the real line $\{\phi_n^\gamma(x)\}_{n\geqslant 0}, \gamma > -1$. These are monic orthogonal polynomials on the interval $[0, +\infty[$ with respect to the weight function $x^\gamma \exp(-x^2)$, $\gamma > -1$, thus satisfying

$$\int_0^{+\infty} x^\gamma \exp(-x^2)\phi_n^\gamma(x)\phi_{n'}^\gamma(x)\, dx = 0,$$

whenever $n \neq n'$.

Note that the half-range generalized Hermite polynomials are related to the Freud polynomials which are orthogonal on $]-\infty, +\infty[$ with respect to the weight function $|x|^\alpha \exp(-x^4)$. These Freud polynomials were studied by Freud [14] and by Nevai [17].

The recurrence formula satisfied by these polynomials is

$$\phi_{n+1}^\gamma(x) = (x - \alpha_n)\phi_n^\gamma(x) - \beta_n\phi_{n-1}^\gamma(x), \quad n\geqslant 0 \tag{4}$$

with

$$\phi_{-1}^\gamma(x) = 0, \quad \phi_0^\gamma(x) = 1.$$

The coefficients α_n and β_n in their turn satisfy the recurrence relations:

$$\beta_n + \beta_{n-1} + \alpha_{n-1}^2 = \frac{2n-1+\gamma}{2}, \tag{5}$$

$$\alpha_n\alpha_{n-1}\beta_n = \left(\frac{n+\frac{\gamma}{2}}{2} - \beta_n\right)^2 - \frac{\gamma^2}{16} \tag{6}$$

with starting values

$$\alpha_0 = \frac{\Gamma\left(\frac{\gamma}{2}+1\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)} \quad \text{and} \quad \beta_0 = 0.$$

The recurrence procedure for the coefficients α_n and β_n appears to be straightforward. Given α_0 and β_0 one calculates β_1 from (5) and then one uses (6) to calculate α_1, and so on.

Unfortunately, while the procedure is simple, the system is rather poorly conditioned. How this comes about and what methods can be used to overcome this problem is explained in [1].

From (5) and (6) we obtain

$$\begin{aligned}
\beta_0 &= 0,\\
\alpha_0 &= \frac{\Gamma\left(\frac{\gamma}{2}+1\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)},\\
\beta_1 &= \frac{\gamma+1}{2} - \left(\frac{\Gamma\left(\frac{\gamma}{2}+1\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)}\right),
\end{aligned}$$

$$\alpha_1 = \frac{\left(\frac{\Gamma\left(\frac{\gamma}{2}+1\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)}\right)^3 - \frac{\gamma}{2}\frac{\Gamma\left(\frac{\gamma}{2}+1\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)}}{\frac{1+\gamma}{2} - \left(\frac{\Gamma\left(\frac{\gamma}{2}+1\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)}\right)^2},$$

etc.

Now the recurrence formula (4) allows us to compute recursively

$$\phi_0^\gamma(x) = 1,$$
$$\phi_1^\gamma(x) = x - \frac{\Gamma\left(\frac{\gamma}{2}+1\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)},$$
$$\phi_2^\gamma(x) = x^2 - \frac{\Gamma\left(\frac{\gamma}{2}+1\right)}{2\Gamma\left(\frac{\gamma+1}{2}\right)\left(\frac{1+\gamma}{2} - \left(\frac{\Gamma\left(\frac{\gamma}{2}+1\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)}\right)^2\right)}x$$
$$+\frac{\left(\Gamma\left(\frac{\gamma}{2}+1\right)\right)^2}{2\left(\Gamma\left(\frac{\gamma+1}{2}\right)\right)^2\left(\frac{1+\gamma}{2} - \left(\frac{\Gamma\left(\frac{\gamma}{2}+1\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)}\right)^2\right)} - \frac{1+\gamma}{2},$$

etc.

We observe that the half-range generalized Hermite polynomials $\{\phi_n^\gamma(x)\}_{n\geqslant 0}$ take the form

$$\phi_n^\gamma(x) = \sum_{k=0}^{n} h_k(\gamma)x^k \quad \text{with } h_n(\gamma) = 1 \quad \text{and} \quad h_k(\gamma) \in \mathbb{R} \;\; k = 0, 1, 2, \ldots, n.$$

In agreement with the general construction theory of Section 3.2, the substitutions

$$\gamma \to m-1 \quad \text{and} \quad x \to i\underline{x},$$

yield the half-range Clifford–Hermite polynomials:

$$\phi_n(i\underline{x}) = \sum_{k=0}^{n} h_k(m-1)(i\underline{x})^k$$

with

$$h_n(m-1) = 1 \quad \text{and} \quad h_k(m-1) \in \mathbb{R} \;\; k = 0, 1, 2, \ldots, n.$$

They satisfy the recurrence relation:

$$\phi_{n+1}(i\underline{x}) = (i\underline{x} - \widetilde{\alpha}_n)\phi_n(i\underline{x}) - \widetilde{\beta}_n\phi_{n-1}(i\underline{x}); \quad n \geqslant 0$$

with

$$\phi_{-1}(i\underline{x}) = 0 \quad \text{and} \quad \phi_0(i\underline{x}) = 1.$$

The coefficients $\widetilde{\alpha}_n$ and $\widetilde{\beta}_n$ in the recurrence relation can be calculated from:

$$\widetilde{\beta}_n + \widetilde{\beta}_{n-1} + \widetilde{\alpha}_{n-1}^2 = \frac{2n+m-2}{2},$$
$$\widetilde{\alpha}_n \widetilde{\alpha}_{n-1} \widetilde{\beta}_n = \left(\frac{n + \frac{m-1}{2}}{2} - \widetilde{\beta}_n \right)^2 - \frac{(m-1)^2}{16},$$

with starting values

$$\widetilde{\alpha}_0 = \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} \quad \text{and} \quad \widetilde{\beta}_0 = 0.$$

A few examples are

$$\phi_0(i\underline{x}) = 1,$$
$$\phi_1(i\underline{x}) = i\underline{x} - \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)},$$
$$\phi_2(i\underline{x}) = (i\underline{x})^2 - \frac{\Gamma\left(\frac{m+1}{2}\right)}{2\Gamma\left(\frac{m}{2}\right)\left(\frac{m}{2} - \left(\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}\right)^2\right)} i\underline{x}$$
$$+ \frac{\left(\Gamma\left(\frac{m+1}{2}\right)\right)^2}{2\left(\Gamma\left(\frac{m}{2}\right)\right)^2\left(\frac{m}{2} - \left(\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}\right)^2\right)} - \frac{m}{2},$$

etc.

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